Heterogeneous populations in a network model of collective motion

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HIGHLIGHTS

• A simplified random network model of collective model with alignment interactions.
• Populations are composed of sub-groups with different external noise.
• The model is analyzed analytically.
• The heterogeneous case is equivalent to an effective homogeneous model.
• The above is not true with spatial segregation.
• The mixed system can become ordered regardless of the noisier populations.

ABSTRACT

Biological systems are typically heterogeneous as individuals vary in their response to the external environment and each other. Here, the effect of heterogeneity on the probability of collective motion is studied analytically in the context of a simple network models of collective motion. We consider a population that is composed of two or more sub-groups, each with different sensitivities to external noise. We find that within a mean-field model, in which particles are sampled uniformly, the dynamics in the heterogeneous case is equivalent to an effective homogeneous model with a noise term that is obtained analytically. This is different than recent simulation results with the well-known Vicsek model of collective motion, in which sub-populations with different noise interact non-additively. In particular, if one of the sub-populations is sufficiently "cold," it dominates the dynamics of the group as a whole. By introducing an ad-hoc bias in the sampling of neighbors, it is shown that the differences between the mean-field and Vicsek dynamics can be explained by the tendency of colder particles to cluster and partially separate from hotter ones. It is analytically shown that, provided the clustering property is sufficiently strong, the mixed system can become ordered regardless of the noisier populations.

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1. Introduction

Complex systems, in particular of biological origins, are typically composed of a heterogeneous population of individuals that may differ in a range of characteristics and properties. In recent years, the effect of heterogeneity on the ability of swarms and other large systems of self-propelled particles to form organized and coherent movement patterns
has been studied both experimentally [1–8] and in simulations [2,7–15]. Past analytical studies on this issue typically concentrated on motility induced phase separation in active Brownian particles or density dependent motility [10,13–18]. In this paper, we study the effect of heterogeneity in the context of the well-studied model proposed by Vicsek et al. [19], in which particles move at a constant speed in a two-dimensional (2D) box with periodic boundaries and locally align their directions with added noise. It has been shown numerically that the system undergoes a phase transition between an ordered phase at low noise/high density and a disordered phase at high noise/low density. Various aspects of the transition have been studied in the rapidly expanding literature on the Vicsek model and its many variants. For recent reviews see [20,21].

Questions regarding heterogeneity within the context of this model has been previously been considered on a mean-field level by introducing variable interactions between two sub-populations [22] or by introducing variability in speeds [23], which was found to only affect the rate of convergence to a stationary distribution but not the distribution itself. In [24], the authors study numerically the order–disorder transition in a modified model in which two sub-populations exist with different added noise. Their main result is that the overall phase of the system is determined by the "colder" population, i.e., the one with lower noise. In particular, the entire system becomes ordered if the smaller noise level (between the two populations) is sufficiently low. This demonstrates that heterogeneity in the Vicsek model can have a non-additive effect on the dynamics.

The main purpose of this manuscript is to address this phenomenon analytically. Since the Vicsek model has no known analytic solution, we analyze instead a simplified version, termed the Vectorial Network Model (VNM), which has been introduced in [25–27] as a mean-field approximation of Vicsek. Note that other forms of mean-field limits, leading to continuum approximations [28,29] are not considered here. Within the VNM, each agent interacts with a random sample of a fixed number of neighbors with added noise. Even though it is highly simplified and discards all spatial features, it could be considered valid for high velocity regimes [25,26], in which particles can travel everywhere in the space in a single time step, causing instantaneous mixing. On the positive side, the simplicity of the VNM allows analytical investigation. Motivated by the works of [25–27], Ariel and Porfiri [30] showed that (in a homogeneous system) the phase of the system is determined by a single parameter — the circular variance of the added noise. This parameter controls whether the disordered state, characterized by a uniform distribution of agent directions, is stable or not. Moreover, it was shown that the circular variance satisfies a fluctuation–dissipation relation, which implies that it can be considered as an effective temperature.

In this paper, the results of [30] are generalized to include two or more sub-populations, each with a different effective temperature. It is shown that the phase of the system is determined by a weighted average of the individual temperatures. Thus, not unexpectedly, the non-additive effect is averaged out. However, introducing a bias in the random sampling of neighbors (as observed in simulation with Vicsek), additivity is lost. In particular, the mixed system becomes ordered if the smallest temperature among the populations is sufficiently low.

The layout of the paper is as follows. Section 2 details the heterogeneous VNM and derives the main results for two sub-populations. Section 3 generalizes our results to biased sampling of neighbors and to any finite number of sub-populations, facilitating comparison with the Vicsek dynamics. We conclude in Section 4.

2. The heterogeneous VNM

We begin with a short review of the homogeneous VNM [25,26]. Definitions and the main derivations are similar to [30].

2.1. Homogeneous VNM

The VNM describes the discrete-time dynamics of $N$ 2D unit vectors $v_1, \ldots, v_N \in \mathbb{C}$. The orientation of the $m$th vector at time $t \in \mathbb{Z}^+ = \{0, 1, \ldots\}$ is denoted $\theta_m(t)$, i.e., $v_m(t) = e^{i \theta_m(t)}$. At every simulation step, each particle aligns with the average direction of $K$ randomly chosen neighbors (with repetitions), which are redrawn independently at every step, plus (intrinsic) noise. More precisely, the dynamics of the VNM is given by

$$\theta_m(t+1) = \arg \{u_m(t)\} + \zeta_m(t),$$

where $\zeta_1(t), \ldots, \zeta_n(t)$ are Independent and Identically Distributed (IID) Random Variables (RVs) in $[-\pi, \pi]$. For simplicity, we consider a symmetric distribution of the noise, i.e., the density of $\zeta_m(t)$, denoted $p_\zeta(\cdot)$, is an even function, $p_\zeta(\theta) = p_\zeta(-\theta)$. Here, $\arg \in [-\pi, \pi]$ is the argument of a complex number, where $\arg(0) = 0$. The vector $u_m(t)$ is the sum of all the vectors accessible by the $m$th vector during the update. Specifically, $u_m(t)$ is

$$u_m(t) = \sum_{j=1}^{K} v_{n_{m,j}(t)},$$

where, for each $m = 1, \ldots, N$, $j = 1, \ldots, K$ and $t \in \mathbb{Z}^+$, $n_{m,j}(t)$ are IID with discrete uniform distribution over $\{1, \ldots, N\}$. The network topology varies in time, eliciting long-range dynamic interactions between the coupled vectors, which enable
Then, the norm of $\Psi(t)$ in [0, 1], whereby $|\Psi(t)| = 1$ for a completely ordered configuration in which all the vectors share a common orientation at the $t$th time step. The limit of the expectation of $|\Psi(t)|$ as $t \to \infty$ is called the polarization. Specifically, we write

$$\text{Pol} = \lim_{t \to \infty} E[|\Psi(t)|],$$

where the expectation is computed with respect to the $\Sigma$-algebra generated by initial conditions, realizations of the noise ($\zeta_m(t)$) and the network evolution ($n_{m,j}(t)$).

In [30] (also reproduced below), it was shown that in the limit of an infinite system, $N \to \infty$, complete disorder, $\theta_m(t)$ IID with density $U(-\pi, \pi)$ for all $m = 1 \ldots N$, is an invariant distribution, i.e., $\theta_m(t+1)$ are also IID with density $U(-\pi, \pi)$. However, this invariant distribution may be either stable or unstable. Denote the effective temperature, $T = 1 - \int_{-\pi}^{\pi} \cos \theta \rho_c(\theta)d\theta$. (5)

We note that $T$ is nothing but an estimator for the circular variance of the centered (circular) random variable for the noise $\zeta_m(t)$ [31]. From the properties of the integral, $0 \leq T \leq 1$, where $T = 0$ for a delta function (at $\theta = 0$) and $T = 1$ is the maximal circular variance obtained with the uniform distribution on $[-\pi, \pi]$. It was shown [30] (also see below) that complete disorder is stable if and only if

$$T > T_c = 1 - \left[ \int_0^{\infty} \int_0^{\infty} J_{m}^{-1}(\lambda) f_1(\lambda) J_1(\lambda) u \lambda d\lambda du \right]^{-1},$$

(6)

where $J_m(\cdot)$ denotes the $m$th Bessel function of the first kind. For example, with $K = 2$, $T_c = 1 - \pi/4$. The threshold $T_c$ is referred to as the critical temperature. If the system’s temperature is above the critical one, $T > T_c$, then the distribution of directions will evolve towards the uniform one, regardless of initial conditions. In this case, it is clear that the polarization vanishes, Pol = 0. Accordingly, we will say that the system is in a disordered phase. However, if $T \leq T_c$, then the system will evolve to a different state, which we do not know how to compute (and may also depend on higher moments of $\zeta$). However, it has a positive polarization, Pol > 0. Accordingly, we will say that the system is in an ordered phase.

The crux of the analytical derivation in [30] lies in understanding that in the thermodynamic limit (fixed time $t$, $N \to \infty$), the vectors $v_1, \ldots, v_N$ become independent. Indeed, Fig. 1 shows that correlations between different particles decay asymptotically as $N^{-1}$. Simulation parameters are $K = 2$ and the noise is uniform in $[-0.2\pi, 0.2\pi]$. A hand waving argument is as follows: up to time $t$, a vector is influenced by at most $tK$ other vectors, which constitute its domain of dependence. Hence, the probability that the $j$th and $k$th vectors do not average over the same vectors is larger than $1 - 2t K/N$, thereby $\text{corr}[v_j(t), v_k(t)] = O(N^{-1})$, where $\text{corr}[v_j(t), v_k(t)] = E[v_j(t)v_k(t)] - E[v_j(t)]E[v_k(t)]$ is the correlation between two arbitrary vectors. Due to the independence property, the VNM process is a discrete time Markov process over a continuum states space, which is the marginal single vector distribution of directions.

2.2. Two sub-populations

For simplicity, we first consider the case in which the agents can be divided into two sub-populations: particles $1, \ldots, N_1$ have a noise distribution with density $\rho_c(\cdot)$, while particles $N_1+1, \ldots, N$ may have a different noise distribution with density $q_c(\cdot)$. We denote by $f_1 = N_1/N$ and $f_2 = 1 - f_1$ the fraction of agents in population 1 and 2, respectively. In addition, we denote

$$T_1 = 1 - \int_{-\pi}^{\pi} \cos \theta \rho_c(\theta)d\theta, \quad T_2 = 1 - \int_{-\pi}^{\pi} \cos \theta q_c(\theta)d\theta,$$

(7)

the effective temperatures of populations 1 and 2.

The derivation below follows [30]. Let $\rho_0(\theta; t)$ and $q_0(\theta; t)$ denote the IID distribution of angles in population 1 and 2, respectively, at time $t$. Then, following the VNM dynamics (1), the distribution of population $i$ at time $t + 1$ is a sum
of two independent variables: the vectorial sum of sampled neighbors with density denoted \( p_u(\cdot; t) \) and the added noise with density \( p_\xi(\cdot) \) or \( q_\xi(\cdot) \). Hence,

\[
p_\theta(\theta; t + 1) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p_u(u, \theta - \xi; t)udq_\xi(\xi)d\xi,
\]

\[
q_\theta(\theta; t + 1) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} p_u(u, \theta - \xi; t)uduq_\xi(\xi)d\xi,
\]

where the density of \( u \in \mathbb{R}^2 \) is written in terms of its size, \( u \in [0, \infty) \) and direction \( \theta \in [-\pi, \pi] \). From (2), \( p_\xi(\cdot; t) \) is a convolution of \( K \) copies of the single vector distributions,

\[
p_u(u, \theta; t) = p_\xi(v, \theta; t) * \cdots * p_\xi(v, \theta; t).
\]

Fig. 1.

Simulation results for the homogeneous VNM with \( K = 2 \) and uniform noise in \([-0.2\pi, 0.2\pi]\). The figure shows the correlations between angles \( \theta \) and between \( v_i = \theta_i - \Psi(t) \) (the entered angle) at two different time steps A. \( t = 10 \) and B. \( t = 20 \). In the thermodynamic limit (fixed \( t, N \to \infty \)), correlations decay asymptotically as \( N^{-1} \).

Source: Reproduced from [30].
where, we used the representation for the Bessel function \[32\],
\[
J_m(x) = \frac{1}{2\pi} \int_{\alpha}^{2\pi+\alpha} e^{i(mx - x^2)} dx.
\]
(15)

A similar expression is obtained for \( \hat{q}_0(n; t + 1) \). Substituting the Fourier decompositions into (2) yields
\[
\hat{p}_u(\lambda, \gamma; t) = \sum_{n=0}^{K} \left( \frac{K}{n!} \right) f_n^0(1 - f_1)^{K-n}[\hat{p}_u(\lambda, \gamma; t)]^{n+1}[\hat{q}_0(\lambda, \gamma; t)]^{K-n}
\]
\[
= \left[ f_1\hat{p}_u(\lambda, \gamma; t) + (1 - f_1)\hat{q}_0(\lambda, \gamma; t) \right]^K,
\]
(16)

where we used Newton’s binomial formula. Writing in terms of the discrete Fourier modes, we used the representation for the Bessel function \[32\],
\[
\hat{p}_u(\lambda, m; t) = (2\pi)^{K-1} \int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} (-i)^n \left[ f_1\hat{p}_u(n; t) + (1 - f_1)\hat{q}_0(n; t) \right] J_n(\lambda)e^{iny} \right\}^K
\]
\[
\times e^{-imy} dy.
\]
(17)

Substituting into (14) yields,
\[
\hat{p}_u(m; t + 1) = (2\pi)^{K-1} \int_{-\pi}^{\pi} \left\{ \sum_{n=-\infty}^{\infty} (-i)^n \left[ f_1\hat{p}_u(n; t) + (1 - f_1)\hat{q}_0(n; t) \right] J_n(\lambda)e^{iny} \right\}^K
\]
\[
\times e^{-imy} dy J_0(u\lambda)u\lambda d\lambda du.
\]
(18)

Similar expressions are obtained for \( \hat{q}_0(m; t + 1) \). Eq. (18) and the analogue one for \( \hat{q} \) completely describe the time-evolution of the model: Given the single-particle distributions at time \( t \), one computes the Fourier transforms. Then, (18) provides the Fourier transforms at time \( t + 1 \), which can be converted to the distributions at time \( t + 1 \) via the inverse transform.

The steady state distribution of the system is given by the fixed points of (18). First, we note the uniform distribution of \([-\pi, \pi]\) is always a fixed point. Substituting in the Fourier decomposition of \( U[-\pi, \pi] \),
\[
\hat{p}_u(m; t) = \hat{q}_0(m; t) = (2\pi)^{-1}\delta_{0,m},
\]
(19)

we readily see it is indeed a fixed point, \( \hat{p}_u(m; t + 1) = \hat{q}_0(m; t + 1) = (2\pi)^{-1}\delta_{0,m} \). The questions is therefore whether this fixed point is stable (indicating a disordered phase) or unstable? For simplicity, we concontinue with the case \( K = 2 \). Generalizations to \( K \geq 3 \) are obtained similar to the treatment in [30]. Substituting \( K = 2 \), expanding the quadratic terms and using the orthogonality of the exponentials, (18) is simplified to
\[
\hat{p}_u(m; t + 1) = (2\pi)^2 \hat{p}_u(m) \times
\]
\[
\sum_{n=-\infty}^{\infty} \left[ f_1\hat{p}_u(n; t) + (1 - f_1)\hat{q}_0(n; t) \right] \left[ f_1\hat{p}_u(m - n; t) + (1 - f_1)\hat{q}_0(m - n; t) \right] J_{m-n}(\lambda).
\]
(20)

where,
\[
I_{m,n} = \int_{0}^{\infty} \int_{0}^{\infty} J_n(\lambda)J_{m-n}(\lambda)J_{m}(u\lambda)u\lambda d\lambda du.
\]
(21)

In [30] it is shown that,
\[
I_{m,n} = \frac{1}{\pi(2n-1)} \sin \left[ \frac{\pi(2n-1)}{2} \right] 2n = m.
\]
(22)

Linearizing (20) around complete disorder (19), the Fourier modes decouple. We obtain, for \( m \neq 0 \),
\[
\begin{pmatrix}
\hat{p}_u(m; t + 1) \\
\hat{q}_0(m; t + 1)
\end{pmatrix}
= 4\pi I_{m,0} \begin{pmatrix}
 f_1\hat{p}_u(m) \\
 f_1\hat{q}_u(m)
\end{pmatrix}
\begin{pmatrix}
(1 - f_1)\hat{p}_u(m) \\
(1 - f_1)\hat{q}_u(m)
\end{pmatrix}
\begin{pmatrix}
\hat{p}_u(m; t) \\
\hat{q}_0(m; t)
\end{pmatrix}.
\]
(23)

Complete disorder is stable if and only if both eigenvalues of the matrix in (23) have a real part with absolute value smaller than 1 for all \( m \neq 0 \) (recall that \( \hat{p}_u(0) = \hat{q}_u(0) = 1/2\pi \)). These eigenvalues are zero and
\[
\lambda_m = 4\pi I_{m,0} \left[ f_1\hat{p}_u(m) + (1 - f_1)\hat{q}_u(m) \right].
\]
(24)
Using (13) and (22),
\[ |\lambda_m| \leq \frac{4}{\pi |m|}. \] (25)

This implies that all modes with $|m| \geq 2$ are stable, regardless of the system parameters. Using again (22) with $m = \pm 1$,
\[ \lambda_1 = 8 \left[ f_1 \hat{p}_c(m) + (1 - f_1) \hat{q}_c(m) \right]. \] (26)

Suppose that $\hat{p}_c(1)$ and $\hat{q}_c(1)$ are non-negative. This holds, for example, if the densities $p_\theta(|x|)$ and $q_\theta(|x|)$ are non-increasing functions of $|x|$. Then, rewriting the condition in terms of the effective temperatures (7), $T_1 = 1 - 2\pi |\hat{p}_\theta(1)|$ and $T_2 = 1 - 2\pi |\hat{q}_\theta(1)|$, we obtain that the system is in the ordered phase if and only if
\[ f_1 T_1 + (1 - f_1) T_2 \leq 1 - \frac{\pi}{4}. \] (27)

This formula is the first main result of this paper. First, taking $f_1 = 1$, it reproduces the result of [30] for the homogeneous case. Moreover, it shows that the phase of mixed two-population systems is determined by a weighted average of the populations. The weights are simply the respective fraction of each population. Fig. 2 shows simulation results with $N = 10^4$ and $f_1 = 0.5$, confirming the analytical derivations. In particular, the solid black line depicts the expected critical temperature predicted by (27). In addition, the level curves shown in Fig. 2 are all with slope -1, demonstrating that the polarization depends on the average temperature. Statistics were obtained using 1000 simulation steps in which the first 10% were discarded.

This result can be generalized to the $K > 2$ case. Repeating the procedure described above we obtain (see [30] for the homogeneous case) that the mixed system is again ordered if and only if
\[ f_1 T_1 + (1 - f_1) T_2 \leq T_c, \] (28)

where $T_c$ is the critical temperature given in the homogeneous case (6).

First, we note that it is imperative to analyze the system in terms of the effective temperatures (i.e., circular variances) rather than other parameters which may be appropriate for homogeneous population, such as the regular variance on $\mathbb{R}$, which is typically used. If both populations have the same type of noise distributions (e.g., both are uniform), then (28) will have the same form written in terms of the variance instead of the effective temperature, $f_1 V_1 + (1 - f_1) V_2 \leq V_c$, where $V_i$ is the variance of the noise of population $i$ and $V_c$ a threshold. However, the if the noise of population 1 has a different shape than that of 2, then (28) does not change, while writing in terms of variances will take a more complicated form in which the symmetry between the two populations may not be apparent.

Second, we find that the effect of heterogeneity in the VNM is additive as the phase of the system is determined by an effective system with the same average temperature. This is in contrast to the dynamics in the Vicsek model, in which the phase is determined by the lower temperature [24]. In that case, numerical simulations suggest that the mixed system is ordered if and only if $\min\{T_1, T_2\}$ is sufficiently small.
Fig. 3. The radial distribution function, $\rho(r)$, describes how density varies as a function of distance from a reference particle. The figure shows $\rho(r)$ for the different populations in the Vicsek model, as a function of the radius from a focal particle, $r$, in units of the interaction distance $R$. Population 1 is colder ($T = 1/3$ and $T = 2/3$). Solid curves show the heterogeneous case: between a test particle and all other particles (red), within population 1 only (blue), within population 2 only (green) and between different populations (purple). The dashed curve shows the radial distribution function of a homogeneous system with the same effective temperature. We see that in the Vicsek model, the average number of interacting particles from the same population is higher compared to the other population. The effect is more pronounced for the colder particles (lower noise).

Source: Reproduced from [24].

This result is intuitively clear since the VNM is essentially a mean field model, i.e., each particle is exposed to the average of the entire population. In order to go beyond the mean-field result, we next consider some simple generalizations to the two-populations case analyzed above.

3. Generalizations

In this section, the analysis presented above is generalized to include biased sampling of neighbors as well as a larger number of sub-populations. These generalizations facilitate a discussion on the differences between the dynamics in the VNM and the Vicsek models.

3.1. Non-uniform mixing

As discussed above, both experiments and theories of SPP systems show that particles with different motility cluster and partially segregate. Indeed, simulations with a heterogeneous Vicsek model, in which each sub-population has a different noise, show that the less motile particles 'stick together' to form larger aggregates. For example, Fig. 3 shows the radial distribution functions between the different populations. Parameters are $f_1 = 0.5$, $\rho = 2$ (average density), $N = 500$, $R = 1$ (interaction distance). The noise terms are uniformly distributed in $[-\pi/3, \pi/3]$ and $[-2\pi/3, 2\pi/3]$. This observation implies that particles in the cold population have a higher probability to align with other particles of the same population compared to the other one.

In order to take this effect into account, we need to change the VNM accordingly. We assume that at every simulation step each particle align with $K$ neighbors sampled independently. However, the probabilities of choosing the population are not necessarily $f_1$ and $1 - f_1$ as in the VNM. Let $\alpha_i$ denote the probability that a particle in population $i$ samples a particle from population $j$. $\alpha_1 + \alpha_2 = 1$. This implies that we now have two versions of the random sum of neighboring vectors, $u_1$ and $u_2$. Eq. (8) changes to

$$p_\theta(\theta; t + 1) = \int_{-\pi}^{\pi} \int_0^{\infty} p_{u_1}(u, \theta - \xi; t)ud\hat{u}_\xi(\xi)d\xi,$$

$$q_\theta(\theta; t + 1) = \int_{-\pi}^{\pi} \int_0^{\infty} p_{u_2}(u, \theta - \xi; t)ud\hat{q}_\xi(\xi)d\xi.$$

As a consequence, Eq. (16) changes to

$$\hat{p}_\nu(\lambda, \gamma; t) = \sum_{n_1=0}^{K} \binom{K}{n_1} \alpha_{11}^{-n_1} \alpha_{21}^{K-n_1} \hat{p}_\nu(\lambda, \gamma; t)^{n_1} [\hat{q}_\nu(\lambda, \gamma; t)]^{K-n_1}$$

$$= [\alpha_{11} \hat{p}_\nu(\lambda, \gamma; t) + \alpha_{21} \hat{q}_\nu(\lambda, \gamma; t)]^K.$$
Propagating these differences forward, complete disorder is still a fixed point. Linearizing around this state yields a multilinear expansion and the derivation proceeds similarly. We obtain the following recursion relation for the $m \neq 0$ Fourier modes,

$$
\begin{pmatrix}
\hat{p}_0(m; t + 1) \\
\vdots \\
\hat{p}_M(m; t + 1)
\end{pmatrix}
= 4\pi I_{m,0}
\begin{pmatrix}
\alpha_{11} \hat{p}_c(m) \\
\vdots \\
\alpha_{22} \hat{q}_c(m)
\end{pmatrix}
\begin{pmatrix}
\hat{p}_0(m; t) \\
\vdots \\
\hat{q}_0(m; t)
\end{pmatrix},
$$

(31)

Taking $\alpha_{11} = f_1$ and $\alpha_{22} = 1 - f_1$, we reproduce the VNM (30).

As a first example, consider the case of two decoupled populations, $\alpha_{11} = \alpha_{22} = 1$. The matrix becomes diagonal and the eigenvalues are

$$
\lambda_1(m) = 4\pi I_{m,0} \hat{p}_c(m), \quad \lambda_2(m) = 4\pi I_{m,0} \hat{q}_c(m).
$$

The system is disordered provided both populations are sufficiently hot. In other words, the system is ordered if and only if

$$
\min\{T_1, T_2\} \leq T_c.
$$

(33)

More generally, we note that the matrix in (31) can be written as

$$
\begin{pmatrix}
\hat{p}_c(m) \\
0
\end{pmatrix}
\begin{pmatrix}
0 & \alpha
\end{pmatrix}
\begin{pmatrix}
\hat{q}_c(m)
\end{pmatrix},
$$

(34)

where $\alpha$ is a stochastic matrix. Hence, the eigenvalues of $\alpha$ have absolute value $\leq 1$ and again, using (13) and (22), we find that all the modes are stable, except possibly $m = \pm 1$. In the latter case, the eigenvalues are $\lambda_1 = 4\pi I_{m,0} x_\pm$, where $x_\pm$ satisfies the quadratic equation,

$$
x^2 - x[\alpha_{11} \hat{p} + \alpha_{22} \hat{q}] + [(\alpha_{11} + \alpha_{22}) \hat{p} - 1] = 0,
$$

(35)

where, for brevity, we write $\hat{p}$ for $\hat{p}_c(1)$ and similarly for $\hat{q}$. To continue, note that since $0 \leq |\hat{p}|, |\hat{q}| \leq (2\pi)^{-1}$ and $0 \leq \alpha_{11}, \alpha_{22} \leq 1$, the last term is negative and the equation has two real solutions. The larger one (in absolute value), $x_+$, is larger than $\alpha_{11} \hat{p} + \alpha_{22} \hat{q}$. With $K = 2$, the disordered state is stable if and only if $8x_+ < 1$. Hence, if $8\alpha_{11} \hat{p} \geq 1$, then the system is in the ordered state regardless of $\alpha_{22}$ and $\hat{q}$. In terms of the effective temperatures, we obtain a sufficient condition for the ordered phase

$$
T_1 < 1 - \frac{1}{\alpha_{11} 4},
$$

(36)

and similarly for $T_2$. This suggests that, provided $\alpha_{11}$ is not too small, there exists an additional critical temperatures $T_{\text{mix}}^j = 1 - \pi/(4\alpha_{11})$ such that if

$$
T_1 \leq T_{\text{mix}}^1 \quad \text{or} \quad T_2 \leq T_{\text{mix}}^2,
$$

(37)

then the system is ordered. Fig. 4 shows simulation results with $\alpha_{11} = \alpha_{22} = 0.9, N = 10^4$ and $f_1 = 0.5$, confirming the theoretical predictions. In particular, we see that if one of the temperatures is below $T_{\text{mix}}$, then the system is in the ordered phase. At higher temperatures, the phase is determined by the weighted average as given by (27). Statistics were obtained using 1000 simulation steps in which the first 10% were discarded. The conclusion of this analysis is that the interesting, non-additive critical behavior observed in the Vicsek model can be explained by the non-homogeneous spatial distribution of the two species.

### 3.2. Many sub-populations

The analysis presented in Section 2 can be repeated with $M > 2$ sub-populations. In this general case, (16) becomes a multilinear expansion and the derivation proceeds similarly. Complete disorder is again a fixed point. With $K = 2$, linearizing around this state yields

$$
\begin{pmatrix}
\hat{p}_0^j(m; t + 1) \\
\vdots \\
\hat{p}_M^j(m; t + 1)
\end{pmatrix}
= 4\pi I_{m,0}
\begin{pmatrix}
\hat{p}_0^j(m) \\
\vdots \\
\hat{q}_c^j(m)
\end{pmatrix}
\begin{pmatrix}
\hat{p}_0^j(m; t) \\
\vdots \\
\hat{q}_0^j(m; t)
\end{pmatrix},
$$

(38)

where $\hat{p}_0^j(m; t)$ denotes the $m$'th Fourier coefficient of the marginal density of orientations of particles in population $j$ at time $t$, $\hat{p}_0^j(m)$ is the $m$'th Fourier coefficient of the noise added to particles in population $j$ and $\alpha$ is an $M \times M$ stochastic matrix whose components $\alpha_{ij}$ denote the probability that a particle in population $i$ samples a particle from population $j$. Similar to the $M = 2$ case, since $\alpha$ is a stochastic matrix, all modes with $|m| \geq 2$ are stable. This only leaves studying the $m = \pm 1$ case.

If the populations are decoupled, $\alpha_{ij} = \delta_{ij}$, the Kronecker delta function and the matrix in (38) is diagonal. Written in terms of the effective temperatures, the system is ordered if and only if

$$
\min\{T_1, \ldots, T_M\} \leq T_c.
$$

(39)
Fig. 4. Simulation results for a heterogeneous biased VNM with $10^4$ particles, $f_1 = 0.5$ and using uniform noise distributions. Each population has a 0.9 probability to sample neighbors from its own population. A. Phase diagram showing the ordered (red dots) and disordered (blue circles) phases. B. The polarization. If the effective temperature of the colder system is sufficiently low, then the entire mixed system is in the ordered phase. The solid lines show the theoretical transition curves — the sloped curve is the critical temperature for the effective heterogeneous system, Eq. (27). The vertical and horizontal lines are the effective temperature for the mixtures, $T_{\text{mix}}$, given by Eq. (36).

At the other end of the spectrum lies the VNM in which all agents have equal probabilities to be sampled, which implies that $\alpha_{ij} = f_j$, the fraction of the $j$'s population. In this case, with $m = 1$, the matrix in (38) can be written as,

$$
\begin{pmatrix}
  f_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  f_M & \cdots & f_M
\end{pmatrix}
\begin{pmatrix}
  \hat{p}_{1\zeta}(1) & \cdots & \hat{p}_{1\zeta}(1) \\
  \vdots & \ddots & \vdots \\
  \hat{q}_{M\zeta}(1) & \cdots & \hat{q}_{M\zeta}(1)
\end{pmatrix}
$$

(40)

It is straightforward that the rank of the matrix on the right is 1. Hence, if $f_1, \ldots, f_M > 0$, then the product has $M - 1$ zero eigenvalues. It is easily verified that the remaining eigenvalue is $\sum_i f_i \hat{p}_{i\zeta}(1)$ with a left eigenvector ($\hat{p}_{1\zeta}(1), \ldots, \hat{p}_{M\zeta}(1)$). Written in terms of the effective temperatures, the system is ordered if and only if

$$
\sum_i f_i \hat{p}_{i\zeta}(1) \leq T_c = 1 - \frac{\pi}{4}.
$$

(41)

Generalizations for $K > 2$ are done similarly.

4. Conclusion

The order–disorder phase transition has been studied in the context of a simplified network model of collective motion with heterogeneous populations, differing in the amount of added noise. It has been shown that in the Vectorial Network Model, in which particles are sampled uniformly, the dynamics in the heterogeneous case is equivalent to an effective homogeneous model with a noise term that was obtained analytically. This result is not surprising considering the mean-field nature of the VNM.

This is in contrast with recent simulation results with the Vicsek model [24], in which the phase of the mixed system is determined only by the colder sub-population. By introducing an ad-hoc bias in the way particles sample their neighbors in the VNM, we can show that the differences between the VNM and Vicsek dynamics can be explained by the tendency of colder particles to cluster and partially separate from the hotter populations. Indeed, we analytically show that, provided the clustering property is strong enough, the mixed system can become ordered regardless of the larger temperature.

On one hand, this work shows the limits of mean-field approaches in studying models of self-propelled particles in which the feedback between orientations and spatial distributions cannot be neglected. On the other hand, our results point out to the precise mechanisms that cause this deviation from the mean-field approximation. However, linking between the biased VNM and Vicsek is far from trivial because the dependency of sub-populations to cluster (in other order, finding the matrix $\alpha$) generally depends of $T_1$ and $T_2$.

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References


